Adaptive Inference Techniques for Some Irregular Problems

Inference! Inference! Inference!

Arun Kumar Kuchibhotla

7 March, 2025

Carnegie Mellon University

Collaborators

Manit Paul (UPenn)

Kenta Takatsu (CMU)

Woonyoung Chang (CMU)

Selina Carter (CMU)

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[1] Joint work with Kenta Takatsu (arXiv:2501.07772)
[2] Joint work with Woonyoung Chang (arXiv:2407.12278)

Motivation and Examples

Inference: confidence intervals

- ⋆ Statistical inference is the cornerstone of statistics and is a necessary ingredient in any rigorous scientific study.
- ⋆ Suppose we have a (real-valued) functional $\theta(P), P \in \mathcal{P}$, e.g., the mean of P or a coefficient in a regression model.
- ⋆ Traditional inference methods such as Wald or resampling (e.g. bootstrap or subsampling) proceed as follows.
- ⋆ Assuming the existence of an estimator $\widehat{\theta}_n$ based on n observations such that

$$r_n(\widehat{\theta}_n - \theta(P)) \xrightarrow{d} L,$$

a confidence interval can be constructed as

$$\widehat{\mathrm{CI}}_{n,\alpha} := \left[\widehat{\theta}_n - \frac{\widehat{q}_{1-\alpha/2}}{\widehat{r}_n}, \widehat{\theta}_n + \frac{\widehat{q}_{\alpha/2}}{\widehat{r}_n}\right],$$

where $\widehat{q}_\gamma$ represents an estimate of the γ-th quantile of the random variable L, and $\widehat{r}_n$ is an estimate of r_n, if unknown.

Limitations of Traditional Inference

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  - Stochastic Gradient Descent
  - Quantile Regression

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- ⋆ The rate of convergence of the estimator can depend on the underlying data generating process.
  - Quantile Regression
  - Monotone Regression
- ⋆ The limiting distribution may be intractable, and the estimator is unstable.
  - Linear Regression
  - Manski's discrete choice model
  - Monotone regression
- ⋆ Finally, traditional methods can be computationally expensive.

Motivating Example 1: Linear Regression

- ⋆ Suppose $(X_i, Y_i), 1 \leq i \leq n$ are IID random vectors with $X_i \in \mathbb{R}^d$. Consider

$$\theta(P) := \arg\min_{\theta \in \mathbb{R}^d} \mathbb{E}[|Y - X^\top \theta|^2].$$

- ⋆ The OLS estimator $\widehat{\theta}_n$ satisfies

$$\|\widehat{\theta}_n - \theta(P)\| = O_p(\sqrt{d/n}), \quad \text{if} \quad d = o(n),$$

but for some matrices Σ, V,

$$n^{1/2}(\widehat{\theta}_n - \theta(P)) \overset{d}{\approx} N(0, \Sigma^{-1} V \Sigma^{-1}), \quad \text{only if} \quad d = o(n^{1/2}).$$

This implies the validity of traditional inference if $d = o(n^{1/2})$.

- ⋆ Most of the results and methods fail if $d \gg n^{1/2}$, because

$$n^{1/2}(c^\top \widehat{\theta}_n - c^\top \theta(P)) \overset{p}{\to} \infty.$$

- ⋆ It is possible to construct a $n^{1/2}$-consistent estimator if $d = o(n^{2/3})$; Chang, Kuchibhotla, and Rinaldo (2023).

Motivating Exampe 2: Stochastic Gradient Descent

- ⋆ Suppose

$$\theta(P) := \arg\min_{\theta \in \Theta} \mathbb{E}_P[m(\theta, Z)].$$

- ⋆ Consider the SGD iterates:

$$\theta^{(t)} = \theta^{(t-1)} - \eta_t \nabla m(\theta^{(t-1)}, Z_t).$$

- ⋆ Polyak and Juditsky proved that

$$n^{1/2}(\overline{\theta}_n - \theta(P)) \overset{d}{\to} N(0, V(P)),$$

  for some variance matrix $V(P)$ that depends on $P, \theta(P)$, and some derivatives of m.

- ⋆ In batch settings, estimating $V(P)$ is not considered hard. But with a computationally efficient algorithm like SGD, it is difficult.

- ⋆ If dimension is "large" compared to n, then no limiting distribution result is available, in general.

Motivating Example 3: Quantile Regression

- Suppose $Y_i = X_i^\top \theta_0 + \xi_i$ are IID such that $\mathrm{Med}(\xi_i | X_i) = 0$.
- If $F_X(t) = \mathbb{P}(\xi \le t | X)$, and for some $\gamma > 0$, (degree of flatness at 0)

$$\lim_{t \to 0} \frac{|F_X(t) - F_X(0)|}{|t|^\gamma} = A_X,$$

  then with $W \sim N(0, \Sigma)$,

$$n^{1/(2\gamma)}(\widehat{\theta}_n - \theta_0) \overset{d}{\to} \operatorname*{arg\,min}_{u \in \mathbb{R}^d} u^\top W + \frac{2}{\gamma + 1} \mathbb{E}[A_X |u^\top X|^{\gamma+1}].$$

- If $\gamma = 1$, then $A_X = f_\xi(0|X)$ and this reduces to the usual asymptotic normality result. In this case, traditional inference is valid.
- The rate of convergence depends on the (unknown) smoothness of the conditional CDF around 0.

Motivating Example 4: Manski's Discrete Choice Model

- ⋆ Suppose $(X_i, Y_i), 1 \le i \le n$ are IID random vectors with $X_i \in \mathbb{R}^d$, $Y_i \in \{0, 1\}$, from Manski's model:

$$Y_i = \mathbf{1}\{X_i^\top \theta(P) + \xi_i \ge 0\} \quad \text{with} \quad \text{Median}(\xi_i | X_i) = 0.$$

- ⋆ This is a semiparametric generlization of logistic regression and is used in Econometrics for discrete choice models.
- ⋆ Manski's estimator of $\theta(P)$ is

$$\widehat{\theta}_n := \operatorname*{arg\,min}_{\theta \in S^{d-1}} \sum_{i=1}^{n} (Y_i - 1/2)\mathbf{1}\{X_i^\top \theta \ge 0\}.$$

- ⋆ If the conditional density of ξ given X exists and is smooth, then

$$n^{1/3}(\widehat{\theta}_n - \theta(P)) \xrightarrow{d} H \times \operatorname*{arg\,min}_{s \in \mathbb{R}^{d-1}} \mathcal{G}(s) + \frac{s^\top V s}{2},$$

for some mean zero Gaussian process $\mathcal{G}(\cdot)$ and some matrix V.

- ⋆ Wald does not apply, bootstrap is inconsistent, and subsampling is unreliable.

Motivating Example 5: Monotone Regression

- ⋆ Consider $(X_i, Y_i), 1 \leq i \leq n$ from the model $Y_i = f_0(X_i) + \xi_i$ where $f_0(\cdot)$ is non-decreasing.

- ⋆ The LSE is given by

$$\widehat{f}_n = \underset{f:\,\mathrm{non-decreasing}}{\arg\min} \sum_{i=1}^{n} (Y_i - f(X_i))^2.$$

- ⋆ If for some $\gamma > 0$, (degree of flatness at x_0)

$$\lim_{t \to 0} \frac{|f_0(x_0 + t) - f_0(x_0)|}{|t|^{\gamma}} = A,$$

then

$$n^{\gamma/(2\gamma+1)}(\widehat{f}_n(x_0) - f_0(x_0)) \stackrel{d}{\to} B_{x_0,\gamma}\mathbb{C}_{\gamma},$$

where $B_{x_0,\gamma}$ is a constant depending on the density of X and variance of ξ at x_0, and $\mathbb{C}_{\gamma}$ is related to a drifted two-sided Brownian motion.

- ⋆ Wald is not applicable, bootstrap is inconsistent, and subsampling is unreliable.

Inference I: COSI Framework

Inference I: COSI (COnfidence sets using Scale Invariance)

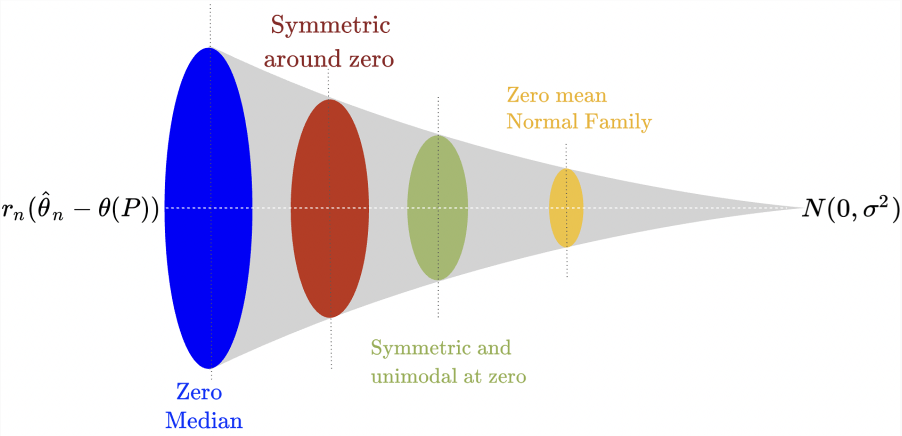

Figure 1: Illustration of Nested Structure of Limiting Distributions

Inference I: COSI Framework

- ⋆ Many estimators have a limiting distribution that meets a scale-invariant property.
- ⋆ A property $\mathfrak{P}$ is called *scale invariant*, if for every random variable W satisfying $\mathfrak{P}$, cW also satisfies $\mathfrak{P}$ for any $c \geq 0$.
- ⋆ Here are a few examples:

Name	Definition
Central Symmetry	$W \overset{d}{=} -W$
Angular Symmetry	$W/\|W\| \overset{d}{=} -W/\|W\|$
Unimodality at 0	$W \overset{d}{=} UZ,\ U \perp Z$
Normal with mean zero	$W \sim N(0, \Sigma)$

- ⋆ Zero is the "center" for any distribution satisfying a scale invariant property.
- ⋆ If $r_n(\widehat{\theta}_n - \theta(P)) \xrightarrow{d} L$ and L satisfies some scale-invariant property, then $\widehat{\theta}_n - \theta(P)$ also *approximately* satisfies the scale-invariant property.

The COSI Algorithm

- Suppose we have n IID observations $Z_1, \ldots, Z_n$.
- Randomly split into B batches of approximately equal size and compute the estimator on each batch. We get

$$\begin{pmatrix} r_{n/B}(\widehat{\theta}^{(1)} - \theta(P)) \\ \vdots \\ r_{n/B}(\widehat{\theta}^{(B)} - \theta(P)) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} L^{(1)} \\ \vdots \\ L^{(B)} \end{pmatrix}.$$

- Note that $L^{(1)}, \ldots, L^{(B)}$ are IID and if the limiting distribution satisfies a scale invariant property $\mathfrak{P}$, then we can think of $\widehat{\theta}^{(j)} - \theta(P), 1 \le j \le B$ as IID observations from a distribution that satisfies $\mathfrak{P}$ approximately.
- Return the confidence set

$$\widehat{\mathrm{CI}}_{n,\alpha} := \left\{\theta : \text{test for } \mathfrak{P} \text{ using } \{\widehat{\theta}^{(j)} - \theta\}_{j=1}^{B} \text{ is not rejected}\right\},$$

- Specific scenarios for univariate functionals to follow.

Scenario I: Zero median

- ⋆ For $\theta(P) \in \mathbb{R}$, consider the scale invariant property of zero median. A random variable $W \in \mathbb{R}$ has zero median if

$$\min\{\mathbb{P}(W \geq 0), \mathbb{P}(W \leq 0)\} \geq 1/2.$$

- ⋆ Asymptotic zero median is same as "estimator is equally likely to over-estimate and under-estimate $\theta(P)$."

- ⋆ A classical test for zero median is the sign test yielding the COSI confidence interval

$$\widehat{\mathrm{CI}}_{n,\alpha}^{\texttt{GHulC}} := \left[\widehat{\theta}^{(\lfloor B/2 \rfloor - c_{B,\alpha})}, \widehat{\theta}^{(\lceil B/2 \rceil + c_{B,\alpha} + 1)}\right], \quad \text{if } B \geq \log_2(2/\alpha).$$

  Here $c_{B,\alpha}$ is the $(1-\alpha/2)$-th quantile of $\mathrm{Bin}(B, 1/2) - \lfloor B/2 \rfloor$.

- ⋆ This is a generalization of the HulC confidence intervals of Kuchibhotla et al. (2024, JRSS-B), studied in Paul and Kuchibhotla (2024).

Scenario II: Symmetry around zero

- ⋆ For $\theta(P) \in \mathbb{R}$, consider the scale invariant property of symmetry around zero. A random variable $W \in \mathbb{R}$ is symmetric around zero if

$$W \overset{d}{=} -W \quad \text{or equivalently} \quad |W| \perp \text{sign}(W).$$

- ⋆ Next to normality, this is the most common case. Quantile regression, Monotone regression, Grenander estimator, and Manski's estimator all satisfy this invariance property.

- ⋆ A classical test for symmetry around zero is the sign-rank test yielding the COSI confidence interval

$$\widehat{\mathrm{CI}}_{n,\alpha}^{\mathtt{Sym}} := \left[A_{\lfloor 2^{B-1}\alpha \rfloor}, A_{2^B - \lfloor 2^{B-1}\alpha \rfloor}\right],$$

  where $A_1 \le A_2 \le \cdots \le A_{2^B-1}$ is the ordered sequence of all subset averages $\{|S|^{-1} \sum_{j \in S} \widehat{\theta}_j : S \subseteq \{1, \ldots, B\}\}$. See Hartigan (1969, JASA) and Maritz (1979, Biometrika).

- ⋆ This yields a generalization of randomization based tests under approximate symmetry of Canay et al. (2017, Econometrica).

Scenario III: Unimodal at zero

- For $\theta(P) \in \mathbb{R}$, consider the scale invariant property of unimodality at zero. A random variable W is unimodal at zero, if

$$W \stackrel{d}{=} UZ \quad \text{for } U \sim \text{Uniform}[0,1], \quad Z \perp U.$$

- Using Edelman's (or Lanke's) confidence interval for mode yields

$$\widehat{\mathrm{CI}}_{n,\alpha}^{\texttt{Mode}} := \left[\widehat{\theta}^{(1)} - t_\alpha(\widehat{\theta}^{(2)} - \widehat{\theta}^{(1)}),\, \widehat{\theta}^{(2)} + t_\alpha(\widehat{\theta}^{(2)} - \widehat{\theta}^{(1)})\right],$$

  with $t_\alpha = (1/\alpha - 1)$.

- This requires only two splits of the data. If more splits are available, one can reduce t_α significantly.

- This is a special case of Unimodal HulC.

- More general confidence intervals for mode are available in the forthcoming paper Paul and Kuchibhotla (2025+).

Finite-sample Micoverage Bounds

- ⋆ For any scale-invariance property $\mathfrak{P}$, we have

$$\mathbb{P}\left(\theta(P) \notin \widehat{\mathrm{CI}}_{n,\alpha}^{\texttt{COSI}}\right) \le \alpha + B \times \max_{1\le j\le B} \mathrm{dist}(\widehat{\theta}^{(j)} - \theta(P), \mathfrak{P}).$$

Finite-sample Micoverage Bounds

- $\star$ For any scale-invariance property $\mathfrak{P}$, we have

$$\mathbb{P}\left(\theta(P) \notin \widehat{\mathrm{CI}}_{n,\alpha}^{\texttt{COSI}}\right) \le {\color{blue}\alpha} + B \times \max_{1 \le j \le B} \mathrm{dist}(\widehat{\theta}^{(j)} - \theta(P), \mathfrak{P}).$$

- $\star$ For zero median property, we have

$$\mathbb{P}(\theta(P) \notin \widehat{\mathrm{CI}}_{n,\alpha}^{\texttt{HulC}}) \le {\color{blue}\alpha} \left(1 + 2(B\Delta)^2 e^{2B\Delta}\right),$$

where

$$\Delta := \max_{1 \le j \le B} \left(\frac{1}{2} - \min_{s \in \{\pm 1\}} \mathbb{P}(s(\widehat{\theta}^{(j)} - \theta(P)) \ge 0)\right)_+ .$$

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- ⋆ For symmetry around zero, we have

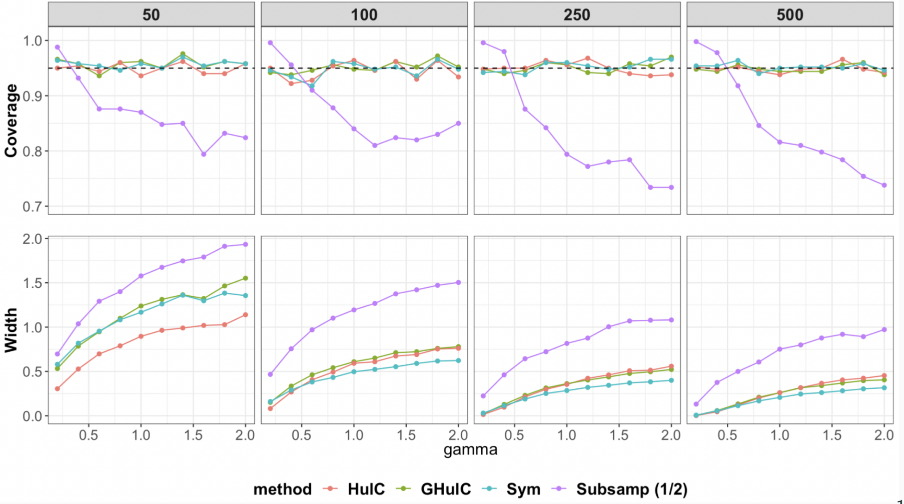
$$\mathbb{P}(\theta(P) \notin \widehat{\mathrm{CI}}_{n,\alpha}^{\texttt{Sym}}) \le {\color{blue}\alpha}\left(1 + 2\Delta\right)^B,$$

where

$$\Delta := \max_{1\le j\le B} \mathbb{E}\left[\left(\frac{1}{2} - \min_{s\in\{\pm1\}} \mathbb{P}(s(\widehat{\theta}^{j} - \theta(P)) \ge 0 \big| |\widehat{\theta}^{(j)} - \theta(P)|)\right)_+\right].$$

Illustration: Quantile Regression

$$Y_i = X_i^\top \beta_0 + \xi_i, X_i \sim N(0, 0.8 I_3 + 0.2 \mathbf{1}\mathbf{1}^\top),$$
$$F_X(t) = 0.5 + 0.5\text{sgn}(t)|t|^\gamma, t \in [-1, 1].$$



Pros and Cons

- ⋆ The procedure needs neither the rate of convergence nor the form of the limiting distribution. It is computationally efficient.
- ⋆ For many scale-invariant properties, finite-sample (or distribution-free) tests can be constructed. This includes central symmetry, angular symmetry, unimodality at zero, and normality with zero mean.
- ⋆ Based on the test used for the scale-invariant property, the resulting confidence sets can have second-order accuracy.
- ⋆ The disadvantage is that one needs to understand the limiting distribution of the estimator to conclude the existence of a scale-invariant property.
- ⋆ This can be difficult, especially in non-parametric or high-dimensional problems (e.g., Lasso or non-parametric regression). Even if one knows the exact limiting distribution, it may not have any scale-invariant property.

Inference II: M-estimation problems[a]

[a] Joint work with Kenta Takatsu (arXiv:2501.07772)

M-estimation Inference

- ⋆ Most functionals encountered in practice can be written as

$$\theta(P) := \arg\min_{\theta \in \Theta} \mathbb{E}_P[m(\theta, Z)],$$

for some loss function $m(\theta, Z)$. OLS, Quantile regression, Manski's model, MLE are some examples.

- ⋆ Setting

$$\mathbb{M}(\theta) := \mathbb{E}_P[m(\theta, Z)],$$

we know that

$$\theta(P) \subseteq \left\{\theta \in \Theta : \mathbb{M}(\theta) \leq \mathbb{M}(\widehat{\theta})\right\},$$

for any estimator $\widehat{\theta} \in \Theta$.

- ⋆ Of course, the right hand set is not computable based on the data. But we can construct two sets based on this intuition and prove their validity.

M-estimation Inference

- Consider

$$\widehat{\mathrm{CI}}_n^{\dagger} := \left\{\theta \in \Theta : \widehat{\mathbb{M}}_n(\theta) - \widehat{\mathbb{M}}_n(\widehat{\theta}_1) \le 0\right\},$$
$$\widehat{\mathrm{CI}}_{n,\alpha} := \left\{\theta \in \Theta : \widehat{\mathbb{M}}_n(\theta) - \widehat{\mathbb{M}}_n(\widehat{\theta}_1) \le \frac{z_{\alpha/2}\widehat{\sigma}(\theta, \widehat{\theta}_1)}{n^{1/2}}\right\}, \quad (1)$$

where $\widehat{\mathbb{M}}_n(\theta) = n^{-1}\sum_{i=1}^{n} m(\theta, Z_i)$ and $\widehat{\theta}_1$ is obtained from an independent sample, and $\widehat{\sigma}(\theta, \widehat{\theta}_1)$ is the sample standard deviation of $m(\theta, Z_i) - m(\widehat{\theta}_1, Z_i), 1 \le i \le n$.

- Clearly,

$$\widehat{\mathrm{CI}}_n^{\dagger} \subseteq \widehat{\mathrm{CI}}_{n,\alpha} \quad \text{for any} \quad \alpha \in (0,1), n \ge 1.$$

- Note that the definition of the confidence sets have no restrictions on Θ or $\widehat{\theta}_1$ except for $\widehat{\theta}_1 \in \Theta$.

- This idea exists in the operations research literature (Vogel (2008, J. of Opt.)) where $\widehat{\theta}_1$ and $\widehat{\mathbb{M}}_n(\cdot)$ are computed on the same data.

Validity

- ⋆ For any $\widehat{\theta}_1$, we have

$$\mathbb{P}(\theta(P) \notin \widehat{\mathrm{CI}}_{n,\alpha}) \le \mathbb{P}(\theta(P) \notin \widehat{\mathrm{CI}}_n^{\dagger}) \le \mathbb{E}\left[\frac{\sigma_P^2(\widehat{\theta}_1)}{\sigma_P^2(\widehat{\theta}_1) + n\mathbb{C}_P^2(\widehat{\theta}_1)}\right],$$

where

$$\sigma_P^2(\theta') := \mathrm{Var}(m(\theta(P), Z) - m(\theta', Z)),$$
$$\mathbb{C}_P(\theta') := \mathbb{E}[m(\theta, Z)] - \min_{\theta \in \Theta} \mathbb{E}[m(\theta, Z)].$$

- ⋆ If $\widehat{\theta}_1$ is consistent for $\theta(P)$, then under mild regularity conditions,

$$\mathbb{P}(\theta(P) \notin \widehat{\mathrm{CI}}_{n,\alpha}) \ge 1 - \alpha - o(1) \quad \text{as} \quad n \to \infty.$$

- ⋆ Neither guarantee depends on Θ or the dimension/definition of $\widehat{\theta}_1$.
- ⋆ With a slight modification, we can obtain finite sample validity for these confidence intervals if the loss is bounded (with a known bound).
- ⋆ Interestingly, we can show that the confidence region $\widehat{\mathrm{CI}}_{n,\alpha}$ shrinks to a singleton at the optimal rate. It adapts!!

Simple, non-trivial example

- ⋆ Consider

$$\theta(P) := \arg\min \mathbb{E}[|Y - X^\top\theta|^2] + h(\theta),$$

where $h(\cdot)$ is some non-stochastic penalty, such as

$$h(\theta) = \quad \lambda\|\theta\|_\rho^\rho, \quad \rho \ge 0 \quad \text{or} \quad \begin{cases} 0, & \text{if } A\theta \le b, \\ +\infty, & \text{if } A\theta \not\le b \end{cases}$$

- ⋆ The OLS would be a penalized/constrained least squares estimator and can be efficiently computed.
- ⋆ However, the limiting distribution of the OLS is incomprehensible because it depends on the derivative of penalty at $\theta(P)$ and/or inequalities that are active at $\theta(P)$, i.e., the coordinates j such that $a_j^\top\theta(P) = b_j$.
- ⋆ To my knowledge, no uniformly valid inference procedure exists except $\widehat{\mathrm{CI}}_{n,\alpha}$. Also, note that our procedure does not require a well-specified linear model.

Inference III: Z-estimation problems[a]

[a] Joint work with Woonyoung Chang (arXiv:2407.12278)

Z-estimation Problems

- ⋆ Z-estimation problems refer to functionals defined as solutions to equations:
$$\mathbb{E}_P[\Psi(\theta(P), Z)] = 0,$$
for some estimating equation $\Psi : \Theta \otimes \mathcal{Z} \to \mathbb{R}^d$ (assuming $\Theta \subseteq \mathbb{R}^d$).
- ⋆ In general, we can consider $\theta(P)$ defined by a set of moment equalities and inequalities. Such weakly/partially identified parameters are common in econometrics.
- ⋆ For any set $\mathcal{A} \subseteq S^{d-1} = \{u \in \mathbb{R}^d : \|u\| = 1\}$, consider the set
$$\widehat{\mathrm{CI}}_{n,\alpha} = \left\{\theta \in \Theta : \sup_{a \in \mathcal{A}} \frac{|\sum_{i=1}^n a^\top \Psi(\theta, Z_i)|}{\sqrt{\sum_{i=1}^n (a^\top \Psi(\theta, Z_i))^2}} \le \kappa_\alpha\right\},$$
where $\kappa_\alpha = \kappa_\alpha(\mathcal{A})$ is the quantile of the maximum of a sequence of Gaussian random variables.

Z-estimation Problems

- ⋆ Validity follows from an application of high-dimensional or infinite-dimensional CLT, and hence, the validity guarantee is tied to the "complexity" of $\mathcal{A}$.

- ⋆ With IID observations, and a bootstrap quantile $\kappa_\alpha(\mathcal{A})$,

$$\sup_{\alpha\in[0,1]} \left|\mathbb{P}(\theta(P) \notin \widehat{\mathrm{CI}}_{n,\alpha}) - \alpha\right| \le \frac{L_4 \log^{5/4}(|\mathcal{A}|)}{n^{1/4}} + \frac{L_q|\mathcal{A}|^{1/q}\log^{3/2}(|\mathcal{A}|)}{n^{1/2-1/q}},$$

where

$$L_q \ := \ \sup_{a\in\mathcal{A}} \frac{(\mathbb{E}|a^\top\Psi(\theta(P), Z)|^q)^{1/q}}{(\mathbb{E}[|a^\top\Psi(\theta(P), Z)|^2])^{1/2}}.$$

- ⋆ If $\mathcal{A} = \{e_j : 1 \le j \le d\}$ and $q = 4$, then validity holds whenever $L_4 < \infty$ and $d = \tilde{o}(n)$.

- ⋆ Note that unlike the procedure for M-estimation problem, no pilot estimator is needed for the construction of the confidence set.

- ⋆ Choosing $\mathcal{A}$ to be a singleton has some interesting implications for a one-dimensional functional of $\theta(P)$.

Application: Linear Regression

- ⋆ Consider

$$\theta(P) \quad \text{satisfying} \quad \mathbb{E}_P[X(Y - X^\top \theta(P))] = 0.$$

- ⋆ Fix any $a \in \mathbb{R}^d$ and consider two sets

$$\widehat{\mathrm{CI}}_{n,\gamma}^{(1)} := \left\{ a^\top \theta : \max_{1 \le j \le d} \frac{|\sum_{i=1}^n e_j^\top \widehat{\Sigma}^{-1} X_i (Y_i - X_i^\top \theta)|}{\sqrt{\sum_{i=1}^n (e_j^\top \widehat{\Sigma}^{-1} X_i (Y_i - X_i^\top \theta))^2}} \le z_{\gamma/(2d)} \right\},$$

$$\widehat{\mathrm{CI}}_{n,\alpha}^{(2)} := \left\{ a^\top \theta : \frac{|\sum_{i=1}^n a^\top \widehat{\Sigma}^{-1} X_i (Y_i - X_i^\top \theta)|}{\sqrt{\sum_{i=1}^n (a^\top \widehat{\Sigma}^{-1} X_i (Y_i - X_i^\top \theta))^2}} \le z_{\alpha/2} \right\}.$$

Then

$$\mathbb{P}\left(a^\top \theta(P) \notin \widehat{\mathrm{CI}}_{n,\alpha}^{(2)} \cap \widehat{\mathrm{CI}}_{n,\gamma}^{(1)} \right) \le \alpha + \gamma \left(1 + \frac{\mathfrak{C} L_3^3 \log^3(2d/\gamma)}{\sqrt{n}} \right),$$

and

$$\mathrm{Width}(\widehat{\mathrm{CI}}_{n,\alpha}^{(2)} \cap \widehat{\mathrm{CI}}_{n,\gamma}^{(1)}) = O_p\left(\frac{1}{\sqrt{n}} + \frac{d}{n} \right).$$

Conclusions

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- ⋆ Estimation has received a lot of focus in both regular and irregular settings.
- ⋆ Traditionally, the construction of tests or confidence sets is mostly based on some estimation procedure and its limiting distribution.
- ⋆ We have discussed three new inference procedures, two of which completely avoid the study of intricate limiting behavior of the pilot estimator.
- ⋆ The validity of all three methods is relatively easy, especially compared to that of resampling methods.
- ⋆ Although the methods are not developed with optimality as a goal, all of them yield optimal adaptive confidence sets.

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Thank You!